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Models for operators with trivial residual space

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1. Introduction

As part of their study of contractions [12], SZ.-NAGY and FOIAŞ derive a functional model for an arbitrary completely non-unitary contraction T, in terms of its characteristic function Θ_T. Also, given an arbitrary purely contractive analytic operator-valued function $\Theta(\lambda)$: $\mathcal{D} \to \mathcal{D}_*$, where $\mathcal{D}$ and $\mathcal{D}_*$ are Hilbert spaces, they are able to construct a model for a completely non-unitary operator whose characteristic function coincides with Θ. The Sz.-Nagy and Foiaş model provides, in fact, a model for the unitary dilation of the contraction, with the model for the contraction itself being obtained by a compression.

In an extension of the dilation theory of Sz.-Nagy and Foiaş, DAVIS [4] has constructed a unitary dilation of an arbitrary operator, with the dilation space being a Krein space, and the dilation preserving the indefinite inner product. In a subsequent paper, DAVIS and FOIAŞ [5] showed how the characteristic function of a noncontraction could be given a geometric interpretation on the dilation space analogous to that used by Sz.-Nagy and Foiaş in their modelling of contractions.

Models have been developed for noncontractions ([1], [3], [8]), which are given in terms of their characteristic functions, but which are along the lines of the DE BRANGES—ROVNYAK model for a contraction [6], providing no model for the dilation. In [10], a model theory is given which does model the dilation space and uses the geometric interpretation of the characteristic function, in a manner analogous to the theory of Sz.-Nagy and Foiaş. As in [5], however, it is necessary in [10] to assume the boundedness of the characteristic function in order to be able to construct this model.

The boundedness of the characteristic function is used in [5] and [10] to ensure the boundedness of the Fourier representations, which map certain subspaces of the dilation space onto L^2 spaces, and to ensure that the characteristic func-

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tion acts as a bounded operator between these spaces. In this paper, we adopt the approach that L^2 spaces are not necessarily the natural ones to be used in modelling noncontractions, and design the function spaces to fit the operator and its characteristic function. Under these circumstances, it is not necessary to assume the boundedness of the characteristic function. Although the model obtained is based on the Sz.-Nagy and Foiaş model of a contraction, the function spaces involved can be defined from the characteristic function in terms of reproducing kernels, and are similar to those considered by de Branges and Rovnyak.

Let T be a bounded operator on a Hilbert space $\mathscr{H}$. Following [12], we write $T\in C_{\cdot 0}$ if $T^{*n}h\to 0$ for all $h\in\mathscr{H}$. When T has bounded characteristic function this condition is equivalent to a condition on the geometry of the dilation space, namely that the residual space be trivial (see [12], [9]). When the characteristic function is not bounded, the condition $T\in C_{\cdot 0}$ implies that the residual space is trivial, but it is possible for an operator not in $C_{\cdot 0}$ to have a trivial residual space.

In this paper, we concentrate on operators with trivial residual space, as the description of the function spaces needed for the model is simplest in this case. The model is also described in terms of an operator valued analytic function Θ, for which we have assumed properties that guarantee that it is the characteristic function of an operator with trivial residual space. The properties assumed for Θ are valid for the characteristic function of an arbitrary $C_{\cdot 0}$ operator; it is not known if only $C_{\cdot 0}$ operators have characteristic functions with these properties.

2. The dilation

In this section, we give a brief description of the Davis dilation of a bounded operator [4].

Using the selfadjoint functional calculus, we can define the operators

$$J_T = \operatorname{sgn}(I-T^*T), \quad Q_T = |I-T^*T|^{1/2},$$
$$J_{T^*} = \operatorname{sgn}(I-TT^*), \quad Q_{T^*} = |I-TT^*|^{1/2}.$$

We have (see [4])

$$J_T Q_T^2 = I-T^*T, \; J_{T^*}Q_{T^*}^2 = I-TT^*,$$
$$TJ_T = J_{T^*}T, \; TQ_T = Q_{T^*}T, \; T^*J_{T^*} = J_TT^*, \; T^*Q_{T^*} = Q_TT^*.$$

We equip the spaces

$$\mathscr{D}_T = J_T\mathscr{H} \quad \text{and} \quad \mathscr{D}_{T^*} = J_{T^*}\mathscr{H}$$

with the respective indefinite inner products

$$[x, y] = (J_T x, y) \quad x, y\in\mathscr{D}_T$$

and

$$[x, y] = (J_{T^*}x, y) \quad x, y \in \mathscr{D}_{T^*},$$

where $(.,.)$ denotes the inner product on $\mathscr{H}$. Then, with the topology inherited from $\mathscr{H}$, $\mathscr{D}_T$ and $\mathscr{D}_{T^*}$ become Krein spaces, with fundamental symmetries J_T and J_{T^*} (see [2]).

The Davis dilation of T is a bounded operator U, which acts on a Krein space $\mathscr{K} \supseteqq \mathscr{H}$, with Hilbert space inner product $(.,.)$ and indefinite inner product $[.,.]$ linked by a fundamental symmetry J:

$$(x, y) = [Jx, y], \quad [x, y] = (Jx, y) \quad \text{for all } x, y \in \mathscr{K}.$$

U is boundedly invertible, and the following properties hold:
(i) $(U^n x, y) = (T^n x, y)$ for all $x, y \in \mathscr{H}$;
(ii) $[Ux, Uy] = [x, y]$ for all $x, y \in \mathscr{K}$;
(iii) $Jx = x$ for all $x \in \mathscr{H}$;
(iv) $\bigvee \{U^n \mathscr{H}: -\infty < n < \infty\} = \mathscr{K}$.

Consider the subspaces

$$\mathscr{L} = \overline{(U-T)\mathscr{H}}, \ \mathscr{L}^* = \overline{(U^*-T^*)\mathscr{H}}, \ \mathscr{L}_* = U\mathscr{L}^* = \overline{(I-UT^*)\mathscr{H}}.$$

(Here, and in the sequel, adjoints are assumed to be taken in the indefinite inner product $[.,.]$ of $\mathscr{K}$.) In the Davis dilation, $\mathscr{L}$ and $\mathscr{L}^*$ are isomorphic to $\mathscr{D}_T$ and $\mathscr{D}_{T^*}$, respectively, in the sense of both the Hilbert space and Krein space structures. There is an operator φ, mapping $\mathscr{L}$ onto $\mathscr{D}_T$, and preserving both the Hilbert space and indefinite inner products, such that

$$\varphi(U-T)h = Q_T h \quad \text{for every } h \in \mathscr{H}. \tag{2.1}$$

As in [12], it is more convenient to work with $\mathscr{L}_*$ than with $\mathscr{L}^*$. We consider an operator φ_*, mapping $\mathscr{L}_*$ onto $\mathscr{D}_{T^*}$, such that $U^*\varphi_*$ preserves both the Hilbert space and indefinite inner products, and such that

$$\varphi_*(I-UT^*)h = J_{T^*}Q_{T^*}h \quad \text{for every } h \in \mathscr{H}.$$

Because of property (ii) above, φ_* also preserves the indefinite inner product.

As in [12], $\mathscr{L}$and $\mathscr{L}^*$ are each orthogonal to $\mathscr{H}$, with respect to both the Hilbert space and the indefinite inner product on $\mathscr{K}$. Consequently, $\mathscr{L}_* = U\mathscr{L}^* \perp U\mathscr{H}$, where we are using "$\perp$" here, and in the sequel, to denote orthogonality with respect to the indefinite inner product. Also, both $\mathscr{L}$ and $\mathscr{L}_*$ are wandering for U, i.e. $U^m \mathscr{L} \perp U^n \mathscr{L}$ and $U^m \mathscr{L}_* \perp U^n \mathscr{L}_*$ for $m \neq n$ (see [7]).

We define

$$\mathscr{K}_+ = \bigvee\{U^n\mathscr{H} : n \geqq 0\},$$
$$M_+(\mathscr{L}) = \bigvee\{U^n\mathscr{L} : n \geqq 0\},$$
$$M_+(\mathscr{L}_*) = \bigvee\{U^n\mathscr{L}_* : n \geqq 0\}.$$

In the Davis dilation, the spaces $\mathscr{H}$ and $M_+(\mathscr{L})$ are mutually orthogonal, in both the Hilbert space and indefinite inner products, and we have

$$\mathscr{K}_+ = \mathscr{H} \oplus M_+(\mathscr{L}).$$

We also have $\mathscr{L}^* \perp \mathscr{K}_+$, and thus

$$\mathscr{L}_* \perp UM_+(\mathscr{L}). \tag{2.2}$$

The *residual space* $\mathscr{R}$ is defined as the space of all vectors in $\mathscr{K}_+$ which are orthogonal to $M_+(\mathscr{L}_*)$ in the indefinite inner product:

$$\mathscr{R} = K_+ \cap M_+(\mathscr{L}_*)^\perp.$$

Theorem 2.1. *If* $T \in C_{\cdot 0}$, *then* $\mathscr{R} = \{0\}$, *i.e.,* $M_+(\mathscr{L}_*) = \mathscr{K}_+$.

Proof. See [9], Theorem 5.5.

Let Q denote the orthogonal projection onto $\mathscr{L}$; in the Davis dilation, this projection is selfadjoint in both inner products. For any $k \in M_+(\mathscr{L})$, the *Fourier coefficients of k in* $M_+(\mathscr{L})$ are defined by

$$l_n = QU^{*n}k, \quad n \geqq 0.$$

The vector k is uniquely determined by its sequence of Fourier coefficients in $M_+(\mathscr{L})$. (See [7].) In the Davis dilation we have, for $k \notin M_+(\mathscr{L})$,

$$\|k\|^2 = \sum_{n=0}^{\infty} \|\varphi l_n\|^2 \quad \text{and} \quad [k, k] = \sum_{n=0}^{\infty} [\varphi l_n, \varphi l_n]. \tag{2.3}$$

The *Fourier representation of k in* $M_+(\mathscr{L})$ is the operator Φ mapping $k \in M_+(\mathscr{L})$ to the function $\Phi k = u$, where

$$u(\lambda) = \sum_{n=0}^{\infty} \lambda^n \varphi l_n$$

and $\{l_n\}$ is the sequence of Fourier coefficients of k in $M_+(\mathscr{L})$. The function u takes its values in $\mathscr{D}_T$ and, because of (2.3), is defined in a neighborhood of zero which includes the open unit disc.

Similarly, for any $k \in M_+(\mathscr{L}_*)$, the *Fourier coefficients of k in* $M_+(\mathscr{L}_*)$ are defined by

$$l_{*n} = PU^{*n}k, \quad n \geqq 0,$$

where P is the orthogonal projection onto $\mathscr{L}_*$: P is selfadjoint in the indefinite inner product, and U^*PU (the orthogonal projection onto $\mathscr{L}^*$) is selfadjoint in both inner products. The structure of $M_+(\mathscr{L}_*)$ can be much more complicated than that described by (2.3) for $M_+(\mathscr{L})$; this will be investigated in subsequent sections. In particular, it is possible to have a vector $k\in M_+(\mathscr{L}_*)$ with the property that $[k,m]=0$ for all $m\in M_+(\mathscr{L}_*)$ (we then call $M_+(\mathscr{L}_*)$ *degenerate*); such a vector k has all of its Fourier coefficients in $M_+(\mathscr{L}_*)$ equal to zero. We will, however, be considering only the case where $M_+(\mathscr{L}_*)$ is nondegenerate, and then any vector in $M_+(\mathscr{L}_*)$ is uniquely determined by its Fourier coefficients. (See [7].)

The *Fourier representation of k in $M_+(\mathscr{L}_*)$* is the operator Φ_* mapping $k\in M_+(\mathscr{L}_*)$ to the function $\Phi_* k=v$, where

$$v(\lambda) = \sum_{n=0}^{\infty} \lambda^n \varphi_* l_{*n}$$

and $\{l_{*n}\}$ is the sequence of Fourier coefficients of k in $M_+(\mathscr{L}_*)$. The function v takes its values in $\mathscr{D}_{T^*}$ and is defined in some neighborhood of zero.

Note that, when $M_+(\mathscr{L}_*)$ is nondegenerate, the Fourier representations Φ and Φ_* are injective, since the Fourier coefficients of a vector k in $M_+(\mathscr{L})$ or in $M_+(\mathscr{L}_*)$ uniquely determine k.

3. The space $H(T)$

Throughout this section we will be assuming that T has spectrum in the closed unit disc. By an application of the principle of uniform boundedness and the spectral radius formula, it follows that the results of this section apply to operators in $C_{\cdot 0}$.

Let us first condider the Kerin space $H^2(\mathscr{D}_T)$ of functions analytic in the open unit disc, with values in $\mathscr{D}_T$, and with square summable Taylor coefficients. If

$$u(\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n \quad \text{and} \quad v(\lambda) = \sum_{n=0}^{\infty} \lambda^n v_n$$

are two functions in $H^2(\mathscr{D}_T)$, then their indefinite and Hilbert space inner products are given by

$$[u, v] = \sum_{n=0}^{\infty} [u_n, v_n] \quad \text{and} \quad (u, v) = \sum_{n=0}^{\infty} (u_n, v_n).$$

A fundamental symmetry J on $H^2(\mathscr{D}_T)$ is given by the formula

$$(Ju)(\lambda) = J_T u(\lambda).$$

The space $H^2(\mathscr{D}_{T^*})$ can be defined in a similar manner.

It follows immediately from (2.3) that the Fourier representation Φ is a unitary operator from $M_+(\mathscr{L})$ onto $H^2(\mathscr{D}_T)$, preserving both the indefinite and the Hilbert space inner products.

When T has bounded characteristic function, every $k \in M_+(\mathscr{L}_*)$ has square summable Fourier coefficients in $M_+(\mathscr{L}_*)$ [5], but for an arbitrary operator, this is not always the case. (See Example 3.1 below, which demonstrates this for a $C_{\cdot 0}$ operator.) Consequently, the Fourier representation Φ_* does not necessarily have its range in the space $H^2(\mathscr{D}_{T^*})$. We will use the notation $H(T)$ to describe the range of Φ_*, i.e.,

$$H(T) = \Phi_* M_+(\mathscr{L}_*).$$

We will assume, for the remainder of this section, that $\mathscr{R} = \{0\}$. We will describe $H(T)$ for such an operator; by Theorem 2.1, we are including the case $T \in C_{\cdot 0}$. The assumption $\mathscr{R} = \{0\}$ is equivalent to $M_+(\mathscr{L}_*) = \mathscr{K}_+$; since $\mathscr{K}_+$ is nondegenerate, it follows that the Fourier representation Φ_* is defined and injective on $\mathscr{K}_+$. Every $k \in \mathscr{K}_+$ has a unique representation of the form $k = h + m$, where $h \in \mathscr{H}$ and $m \in M_+(\mathscr{L})$; thus, since Φ_* is injective, every function in $H(T)$ has a unique representation of the form

$$\Phi_* k = \Phi_* h + \Phi_* m,$$

where $h \in \mathscr{H}$ and $m \in M_+(\mathscr{L})$.

We can define a Krein space structure on $H(T)$ by requiring that Φ_* be unitary with respect to both inner products; we define, for $k, k' \in M_+(\mathscr{L}_*)$,

$$[\Phi_* k, \Phi_* k'] = [k, k'] \tag{3.1}$$

and

$$(\Phi_* k, \Phi_* k') = (k, k'). \tag{3.2}$$

$H(T)$ is then a Krein space, with a fundamental symmetry (also denoted by J) given by

$$J\Phi_* k = \Phi_* J k.$$

We begin our study of the structure of $H(T)$ by considering first the subspace $\Phi_* \mathscr{H}$. For $h \in \mathscr{H}$, define a function Fh by

$$[Fh](\lambda) = \sum_{n=0}^{\infty} \lambda^n J_{T^*} Q_{T^*} T^{*n} h = J_{T^*} Q_{T^*} (I - \lambda T^*)^{-1} h. \tag{3.3}$$

If we apply [9], Corollary 8.2, to the our situation, in which $\mathscr{R} = \{0\}$, we obtain

$$\Phi_* h = Fh$$

for all $h \in \mathscr{H}$. Fh is not necessarily in $H^2(\mathscr{D}_{T^*})$, since the sequence $\{J_{T^*} Q_{T^*} T^{*n} h\}_{n \geq 0}$ need not be square summable, even for $T \in C_{\cdot 0}$:

Example 3.1. Let $\{a_m\}_{m \geq 1}$ be the sequence of positive numbers given by $a_m^2 = 1 - 1/m^2$, and let T_m be an operator on a two dimensional Hilbert space $\mathscr{H}_m$, given by the matrix

$$T_m = \begin{pmatrix} a_m & 1 \\ 0 & 0 \end{pmatrix}.$$

Let $\mathscr{H}$ be the Hilbert space of square summable sequences $x=\{x_m\}_{m\geqq 1}$, with $x_m \in \mathscr{H}_m$ for $m\geqq 1$, and define T on $\mathscr{H}$ by $Tx=\{T_m x_m\}_{m\geqq 1}$. Since $0\leqq a_m<1$, for each $m\geqq 1$, it is easily verified that both T and its adjoint are $C_{\cdot 0}$ operators.

We have $J_{T^*} Q_{T^*} T^{*n} x = \{y_{nm}\}_{m\geqq 1}$, where

$$y_{nm} = \begin{pmatrix} -a_m^{n+1} & 0 \\ a_m^{n-1} & 0 \end{pmatrix} x_m \quad \text{when} \quad n>0$$

and

$$y_{0m} = \begin{pmatrix} -a_m & 0 \\ 0 & 1 \end{pmatrix} x_m.$$

If

$$x_m = \begin{pmatrix} 1/m \\ 0 \end{pmatrix}$$

then we have, for every $M>0$,

$$\sum_{n=0}^{\infty} \|J_{T^*} Q_{T^*} T^{*n} x\|^2 \geqq \sum_{m=1}^{M} m^{-2}\Big(a_m^2 + \sum_{n=1}^{\infty} (a_m^{2n+2} + a_m^{2n-2})\Big) = \tag{3.4}$$

$$= \sum_{m=1}^{M} m^{-2}(1-a_m^2)^{-1}(1+a_m^2) = \sum_{m=1}^{M} (1+a_m^2) \to \infty$$

as $M\to\infty$. Therefore, the sequence $\{J_{T^*} Q_{T^*} T^{*n} x\}_{n\geqq 0}$ is not square summable. It also follows, from the observation made earlier, that T does not have a bounded characteristic function.

As in [12], [5], [9], and [10], the space $\Phi_* M_+(\mathscr{L})$ can be studied by introducing the characteristic function Θ_T of T, defined by

$$\Theta_T(\lambda) = [-TJ_T + \lambda J_{T^*} Q_{T^*}(I-\lambda T^*)^{-1} J_T Q_T] | \mathscr{D}_T. \tag{3.5}$$

$\Theta_T(\lambda)$ is defined for those complex numbers λ for which $I-\lambda T^*$ is boundedly invertible, and takes values which are bounded operators from $\mathscr{D}_T$ to $\mathscr{D}_{T^*}$. Since the spectrum of T is in the closed unit disc, it follows that $\Theta_T(\lambda)$ is defined for λ in the open unit disc. We can write, for $|\lambda|<1$,

$$\Theta_T(\lambda) = \sum_{n=0}^{\infty} \lambda^n \Theta_n, \tag{3.6}$$

where

$$\Theta_0 = -TJ_T \tag{3.7}$$

and

$$\Theta_n = J_{T^*} Q_{T^*} T^{*n-1} J_T Q_T \tag{3.8}$$

for $n\geqq 1$.

The characteristic function Θ_T is purely contractive, i.e., if $|\lambda|<1$, then

$$[\Theta_T(\lambda)a, \Theta_T(\lambda)a] < [a, a] \quad \text{for} \quad a\in\mathscr{D}_T,\ a\neq 0,$$

and

$$[\Theta_T(\lambda)^* b, \Theta_T(\lambda)^* b] < [b, b] \quad \text{for} \quad b\in\mathscr{D}_{T^*},\quad b\neq 0.$$

As usual, we are using $[.,.]$ to denote the indefinite inner products on $\mathscr{D}_T$ and $\mathscr{D}_{T^*}$, and $\Theta_T(\lambda)^*$ denotes the adjoint of $\Theta_T(\lambda)$ with respect to these inner products.

When $T\in C_{\cdot 0}$, we also have, in the strong operator topology, the telescoping series

$$\sum_{n=0}^{\infty}\Theta_n^*\Theta_n = J_T T^* T J_T + \sum_{n=1}^{\infty} Q_T T^{n-1}(I-TT^*)T^{*n-1}J_T Q_T = \tag{3.9}$$

$$= T^*T + Q_T(I-\lim_{n\to\infty} T^n T^{*n})J_T Q_T = I.$$

Suppose $u\in H^2(\mathscr{D}_T)$, and consider the function $\Theta_T u$, defined for $|\lambda|<1$ by $[\Theta_T u](\lambda)=\Theta_T(\lambda)u(\lambda)$. If Θ_T is uniformly bounded on the open unit disc, then $\Theta_T u\in H^2(\mathscr{D}_{T^*})$ and the Fourier representation Φ_* maps $M_+(\mathscr{L}_*)$ onto $H^2(D_{T^*})$; if, in addition, $\mathscr{R}=\{0\}$, then we have, for $m\in M_+(\mathscr{L})$,

$$\Theta_T\Phi m = \Phi_* m \tag{3.10}$$

(see [5]).

We have already noted that Example 3.1 gives an example of a $C_{\cdot 0}$ operator whose characteristic function is not bounded. Indeed $\Theta_T u\notin H^2(\mathscr{D}_{T^*})$ for the operator T of Example 3.1 and for $u(\lambda)$ equal to the constant function whose range is the vector x of Example 3.1. We can still, nevertheless, generalise (3.10) to an arbitrary operator having $\mathscr{R}=\{0\}$:

Theorem 3.1. *If $\mathscr{R}=\{0\}$, then $\Phi_* m=\Theta_T\Phi m$ for all $m\in M_+(\mathscr{L})$.*

Proof. Since the proof in [5] relies on Θ_T being a bounded operator into $H^2(\mathscr{D}_{T^*})$, it can not be used here. The proof given here does not require that the dilation be the one constructed by Davis, but only that the operators $\varphi\colon \mathscr{L}\to\mathscr{D}_T$ and $\varphi_*\colon \mathscr{L}_*\to\mathscr{D}_{T^*}$, defined in Section 2 above, be bounded.

Let us denote by P the projection onto $\mathscr{L}_*$ which is selfadjoint with respect to the indefinite inner product, and, as usual, let "$\perp$" denote orthogonality with respect to the indefinite inner product.

Suppose $h\in\mathscr{H}$, and let $m=(U-T)h\in\mathscr{L}$, so that $\varphi m=Q_T h$. Since $\mathscr{L}_*\perp U\mathscr{H}$, and since we have

$$m = (U-T)h = -(I-UT^*)Th + U(I-T^*T)h\in\mathscr{L}_* + U\mathscr{H},$$

we can conclude that

$$Pm = -(I-UT^*)Th.$$

Consequently,

$$\varphi_* Pm = -J_{T^*}Q_{T^*}Th = -TJ_T Q_T h = -TJ_T \varphi m \tag{3.11}$$

for a set of vectors m which are dense in $\mathscr{L}$. (3.11) extends by continuity to be valid for all $m \in \mathscr{L}$.

We also have, for all $n>0$ and for $m=(U-T)h$ ($h \in \mathscr{H}$), the telescoping series

$$U^{*n}m = U^{*n}(U-T)h = \tag{3.12}$$

$$= -U^{*n}(I-UT^*)Th + \sum_{k=0}^{n-1} U^{*n-1-k}(I-UT^*)T^{*k}(I-T^*T)h + UT^{*n}(I-T^*T)h.$$

Since $\mathscr{L}_*$ is wandering for U, and $U\mathscr{H} \perp \mathscr{L}_*$, all except one of the terms in (3.12) are orthogonal to $\mathscr{L}_*$ (in the indefinite inner product), and we can conclude that

$$PU^{*n}m = (I-UT^*)T^{*n-1}(I-T^*T)h.$$

Consequently,

$$\varphi_* PU^{*n}m = J_{T^*}Q_{T^*}T^{*n-1}(I-T^*T)h = J_{T^*}Q_{T^*}T^{*n-1}J_T Q_T \varphi m \tag{3.13}$$

for all $n>0$ and for a set of vectors m which are dense in $\mathscr{L}$. Again, (3.13) extends by continuity to be valid for all $m \in \mathscr{L}$.

If we use the representation (3.6) of $\Theta_T(\lambda)$, then (3.11) and (3.13) can be rewritten

$$\varphi_* PU^{*n}m = \Theta_n \varphi m, \tag{3.14}$$

for all $m \in \mathscr{L}$ and $n \geqq 0$.

Now suppose $m \in M_+(\mathscr{L})$, and let $\{l_n\}_{n\geqq 0}$ be the sequence of Fourier coefficients of m in $M_+(\mathscr{L})$. Then we have, for each $n \geqq 0$,

$$m - \sum_{k=0}^{n} U^k l_k \in U^{n+1}M_+(\mathscr{L})$$

(see [7]), and thus

$$U^{*n}m - \sum_{k=0}^{n} U^{*n-k} l_k \in UM_+(\mathscr{L}). \tag{3.15}$$

We have $UM_+(\mathscr{L}) \perp \mathscr{L}_*$ (see (2.2)), and thus (3.15) and (3.14) imply that

$$\varphi_* PU^{*n}m = \sum_{k=0}^{n} \varphi_* PU^{*n-k} l_k = \sum_{k=0}^{n} \Theta_{n-k}\varphi l_k. \tag{3.16}$$

Since we are assuming that $\mathscr{R}=\{0\}$, the vector m is in $M_+(\mathscr{L}_*)$, and the left side of (3.16) is the coefficient of λ^n in the Taylor expansion of $\Phi_* m$. The right side of (3.16) is the coefficient of λ^n in the Taylor expansion of $\Theta_T \Phi m$. Therefore we have $\Phi_* m = \Theta_T \Phi m$, and the theorem is proved.

Corollary 3.3. *If* $\mathcal{R}=\{0\}$, *then* $\Theta_T u \in H(T)$ *for all* $u \in H^2(\mathcal{D}_T)$.

4. Some properties of $H(T)$

In this section, we derive some properties of $H(T)$ that will be useful in constructing a model for T in terms of its characteristic function. As before, we will be assuming throughout this section that T has spectrum in the closed unit disc and that $\mathcal{R}=\{0\}$, but some results will be proved only for $C_{\cdot 0}$ operators.

It follows from the results of the preceding section that, if $\mathcal{R}=\{0\}$, then the range of the Fourier representation Φ_* is of the form

$$H(T) = F\mathcal{H} + \Theta_T H^2(\mathcal{D}_T).$$

If a vector $k \in \mathcal{K}_+$ is written in the form $k=h+m$, with $h \in \mathcal{H}$ and $m \in M_+(\mathcal{L})$, then we have

$$\Phi_* k = Fh + \Theta_T u, \tag{4.1}$$

where $u = \Phi m$. The representation (4.1) of a function in $H(T)$ is unique, since Φ and Φ_* are injective. The inner products on $H(T)$ have a simple formulation in terms of the representation (4.1):

Proposition 4.1. *If* $v = Fh + \Theta_T u$ *and* $v' = Fh' + \Theta_T u'$ *are two functions in* $H(T)$, *then*

$$[v, v'] = (h, h') + [u, u'] \tag{4.2}$$

and

$$(v, v') = (h, h') + (u, u'). \tag{4.3}$$

Proof. Let $v = \Phi_* k$ and $v' = \Phi_* k'$, where $k = h+m$ and $k' = h'+m'$. It follows immediately from (4.1) and the definitions of the inner products (3.1) and (3.2) on $H(T)$ and on $\mathcal{K}$ (see [4]) that

$$[v, v'] = [k, k'] = (h, h') + [m, m'] = (h, h') + [u, u'],$$

since Φ is a unitary operator. The formula for (v, v') is proved similarly.

It is important for a later application to note that uniqueness of the representation (4.1) in fact implies the condition $\mathcal{R}=\{0\}$.

Theorem 4.2. *Suppose T is an operator with spectrum in the closed unit disc. Define the operator-valued functions F and Θ_T by* (3.3) *and* (3.5), *respectively. If*

$\mathcal{R}\neq\{0\}$, then there exists a vector $h\in\mathcal{H}$ and a function $u\in H^2(\mathcal{D}_T)$, not both zero, such that $Fh+\Theta_T u=0$.

Proof. If $\mathcal{R}\neq\{0\}$, then there is a nonzero vector $k\in\mathcal{R}$, i.e., $k\in\mathcal{K}_+$ and $k\perp M_+(\mathcal{L}_*)$. We can write k in the form $k=h+m$, where $h\in\mathcal{H}$ and $m\in M_+(\mathcal{L})$, and we will take $u=\Phi m\in H^2(\mathcal{D}_T)$. Since $k\neq 0$, h and u are not both zero. If $\{l_n\}_{n\geqq 0}$ is the sequence of Fourier coefficients of m in $M_+(\mathcal{L})$, then we have, for the nth coefficient in the Taylor series expansion of u,

$$u_n = \varphi l_n. \tag{4.4}$$

Since $k\in\mathcal{R}$, we can apply [9], Theorem 4.2, and assert the existence of a sequence $\{h_n\}_{n\geqq 0}$ of vectors in $\mathcal{H}$ such that

$$h_0 = h, \tag{4.5}$$

$$Th_{n+1} = h_n \quad \text{for all} \quad n\geqq 0, \quad \text{and} \tag{4.6}$$

$$l_n = (U-T)h_{n+1} \quad \text{for all} \quad n\geqq 0. \tag{4.7}$$

Combining (4.4) with (4.7) and (2.1) gives us

$$u_n = Q_T h_{n+1} \quad \text{for all} \quad n\geqq 0. \tag{4.8}$$

The nth coefficient v_n in the Taylor series expansion of $v=Fh+\Theta_T u$ can now be calculated. From the definitions of F and Θ_T we obtain

$$v_n = J_{T^*}Q_{T^*}T^{*n}h+\sum_{m=0}^{n}\Theta_m u_{n-m} =$$

$$= J_{T^*}Q_{T^*}T^{*n}h_0 - TJ_TQ_Th_{n+1}+\sum_{m=1}^{n}J_{T^*}Q_{T^*}T^{*m-1}(I-T^*T)h_{n-m+1}$$

by (4.5) and (4.8). The second term of the above line can be written as $-J_{T^*}Q_{T^*}Th_{n+1}$, and iterating (4.6) gives us $h_{n-m+1}=T^m h_{n+1}$ $(1\leqq m\leqq n+1)$. Thus we have

$$v_n = J_{T^*}Q_{T^*}\Big(T^{*n}T^{n+1}-T+\sum_{m=1}^{n}T^{*m-1}(I-T^*T)T^m\Big)h_{n+1} = 0,$$

since the series telescopes. Thus $v=0$, and the theorem is proved.

We will denote by $\mathbf{U}$ the operator of multiplication by the independent variable on $H(T)$ or $H^2(\mathcal{D}_T)$:

$$[\mathbf{U}u](\lambda) = \lambda u(\lambda)$$

for $u\in H(T)$ or $u\in H^2(\mathcal{D}_T)$. It is obvious that $H^2(\mathcal{D}_T)$ is invariant for $\mathbf{U}$, and that $\mathbf{U}$ preserves both the Hilbert space and the indefinite inner products of $H^2(\mathcal{D}_T)$. The adjoint $\mathbf{U}^*$ of $\mathbf{U}$, in both inner products of $H^2(\mathcal{D}_T)$, is given by the formula

$$[\mathbf{U}^*u](\lambda) = \lambda^{-1}\big(u(\lambda)-u(0)\big). \tag{4.9}$$

$H(T)$ is the range of the Fourier representation Φ_*, and the inner products on $H(T)$ were defined so as to make Φ_* unitary. It follows easily from the definition of Φ_* that

$$\Phi_* U = \mathbf{U}\Phi_*,$$

where U is the Davis dilation of T, and so $H(T)$ is invariant for $\mathbf{U}$. Since U is bounded and preserves the indefinite inner product of $\mathscr{K}$, we can conclude that $\mathbf{U}$ is bounded and preserves the indefinite inner product of $H(T)$. The formula (4.9) for the adjoint $\mathbf{U}^*$ of $\mathbf{U}$, in the indefinite inner product, is also valid in $H(T)$, since $\mathbf{U}^*$ acts as the backward shift on the Fourier coefficients of a vector in $M_+(\mathscr{L}_*)$. It is possible to give explicitly the action of $\mathbf{U}$ and $\mathbf{U}^*$ on $H(T)$ in terms of the representation (4.1):

Proposition 4.3. *Suppose $v\in H(T)$, with $v=Fh+\Theta_T u$ for some $h\in\mathscr{H}$ and $u\in H^2(\mathscr{D}_T)$. Then*

$$\mathbf{U}v = FTh+\Theta_T(Q_T h+\mathbf{U}u) \tag{4.10}$$

and

$$\mathbf{U}^* v = F(T^* h+J_T Q_T u(0))+\Theta_T(\mathbf{U}^* u). \tag{4.11}$$

Proof. It follows immediately from (3.5) that

$$\Theta_T(\lambda)Q_T = J_{T^*}Q_{T^*}(I-\lambda T^*)^{-1}(\lambda-T)$$

(cf. [12], p. 237). Therefore, if $v=Fh+\Theta_T u$, we have

$$\lambda v(\lambda) = \lambda J_{T^*}Q_{T^*}(I-\lambda T^*)^{-1}h+\lambda\Theta_T(\lambda)u(\lambda) =$$
$$= J_{T^*}Q_{T^*}(I-\lambda T^*)^{-1}Th+\Theta_T(\lambda)Q_T h+\lambda\Theta_T(\lambda)u(\lambda).$$

Formula (4.10) follows.

If $v'=Fh'+\Theta_T u'$, for some $h'\in\mathscr{H}$ and $u'\in H^2(\mathscr{D}_T)$, then, using (4.10) and Proposition 4.1, we have

$$[\mathbf{U}^* v, v'] = [v, \mathbf{U}v'] = (h, Th')+[u, Q_T h'+Uu'] = \tag{4.12}$$
$$= (T^* h, h')+[u(0), Q_T h']+[\mathbf{U}^* u, u'].$$

Note that

$$[u(0), Q_T h'] = (J_T u(0), Q_T h') = (J_T Q_T u(0), h'),$$

so that (4.11) follows from (4.12) and Proposition 4.1.

The fact that $\mathbf{U}$ preserves the indefinite inner product of $H(T)$ can be verified directly by observing that (4.10) and (4.11) imply that $\mathbf{U}^*\mathbf{U}=I$.

The inner products (3.1) and (3.2) have been defined on the function space $H(T)$ by making reference to the underlying Krein space $\mathscr{K}$. The indefinite inner product (3.1) can also be given, on a dense subset of $H(T)$, directly in terms of the functions involved. In this section, we will prove that, for $v \in H(T)$ and for u belonging to a dense subset of $H(T)$, with

$$u(\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n \quad \text{and} \quad v(\lambda) = \sum_{n=0}^{\infty} \lambda^n v_n,$$

the indefinite inner product (3.1) is also given by

$$[u, v]_\Sigma = \sum_{n=0}^{\infty} [u_n, v_n]. \tag{4.13}$$

The formula (4.13) is identical to the one that applies in $H^2(\mathscr{D}_{T^*})$, but the Hilbert space structure on these two spaces can be quite different.

The dense subset of $H(T)$ on which (4.13) is valid includes the polynomials and a space of functions obtained from a reproducing kernel for the indefinite inner product. When $T \in C_{\cdot 0}$, (4.13) is also valid for any functions u and v which are finite linear combinations of functions of the form $\mathbf{U}^n Fh$ $(h \in \mathscr{H}, n \geqq 0)$, and for any u and v of the form $Fh + \Theta_T p$, where $h \in \mathscr{H}$ and p is a polynomial in $H^2(\mathscr{D}_T)$.

Example 5.9 of [9] shows that (4.13) can not be expected, in general, to provide the indefinite inner product on all of $H(T)$. In this example, we have $\mathscr{R} = \{0\}$, but $T \notin C_{\cdot 0}$. It is a consequence of Proposition 4.6 below that $[Fh, Fh']_\Sigma \neq (h, h')$ for some $h, h' \in \mathscr{H}$, and thus (4.13) does not give the inner product on all of $H(T)$. It is not known whether or not (4.13) is valid on all of $H(T)$ when $T \in C_{\cdot 0}$.

The space $H(T)$ contains the constant functions with values in $\mathscr{D}_{T^*}$ (they are the functions of the form $\Phi_* m$ for $m \in \mathscr{L}_*$), and hence contains all polynomials with values in $\mathscr{D}_{T^*}$. The operator mapping a vector in $\mathscr{D}_{T^*}$ to the corresponding constant function is continuous and preserves the indefinite inner product, because of the corresponding properties of the operator φ_* considered in Section 2. Moreover, the definition of $M_+(\mathscr{L}_*)$ and the wandering property of $\mathscr{L}_*$ imply that the polynomials are dense in $H(T)$ and that the indefinite inner product of a polynomial with an arbitrary function in $H(T)$ is given by the formula (4.13). These properties of the polynomials can be verified directly in terms of the representation (4.1) of functions in $H(T)$, given for polynomials in the following proposition. The operators Θ_k in (4.15) are those given by (3.7) and (3.8), and their adjoints are taken in the indefinite inner product.

Proposition 4.4. *If $a \in \mathscr{D}_{T^*}$, then the constant function with range a in $H(T)$ is of the form*

$$a = F(Q_{T^*} a) + \Theta_T(-J_T T^* a). \tag{4.14}$$

We also have, for all $n \geqq 0$,

$$\mathbf{U}^n a = F(T^n Q_{T^*} a) + \Theta_T\Big(\sum_{m=0}^{n} \mathbf{U}^m \Theta_{n-m}^* a\Big). \tag{4.15}$$

If $v \in H(T)$ *and if* p *is a polynomial in* $H(T)$, *then*

$$[p, v] = [p, v]_\Sigma. \tag{4.16}$$

Proof. From the definitions of F and Θ_T we obtain

$$[F(Q_{T^*}a) + \Theta_T(-J_T T^* a)](\lambda) =$$
$$= J_{T^*} Q_{T^*}(I-\lambda T^*)^{-1} Q_{T^*} a + TT^* a - \lambda J_{T^*} Q_{T^*}(I-\lambda T^*)^{-1} T^* Q_{T^*} a =$$
$$= J_{T^*} Q_{T^*}(I-\lambda T^*)^{-1}(I-\lambda T^*) Q_{T^*} a + TT^* a = a,$$

since $J_{T^*} Q_{T^*}^2 = I - TT^*$. Thus, (4.14) is proved, and (4.15) follows by iterating (4.10) and using the definitions (3.7) and (3.8) of Θ_k.

For an arbitrary $v = Fh + \Theta_T u \in H(T)$ we have, from (4.15) and Proposition 4.1,

$$[\mathbf{U}^n a, v] = (T^n Q_{T^*} a, h) + \sum_{m=0}^{n} [\Theta_{n-m}^* a, u_m], \tag{4.17}$$

where u_m is the mth coefficient in the Taylor series expansion of u. Rewriting each of the terms on the right side of (4.17) in the indefinite inner product of $\mathscr{D}_{T^*}$ gives us

$$[\mathbf{U}^n a, v] = [a, J_{T^*} Q_{T^*} T^{*n} h + \sum_{m=0}^{n} \Theta_{n-m} u_m] = [a, v_n],$$

where v_n is the nth coefficient in the Taylor series expansion of v. Formula (4.16) then follows from the definition (4.13) of $[.,.]_\Sigma$ and the linearity of the inner product.

Consider the function

$$k(\mu, \lambda) = (1 - \lambda\bar{\mu})^{-1},$$

defined for λ and μ in the open unit disc, and the associated family $\{k_\mu\}$ of functions of a single variable, defined by

$$k_\mu(\lambda) = k(\mu, \lambda).$$

For any $a \in \mathscr{D}_T$ and $|\mu| < 1$, the function $k_\mu a$ is in $H^2(\mathscr{D}_T)$ and has the reproducing properties:

$$[u, k_\mu a] = [u(\mu), a] \tag{4.18}$$

and

$$(u, k_\mu a) = \big(u(\mu), a\big) \tag{4.19}$$

for every $u \in H^2(\mathscr{D}_T)$. The inner products on the left sides of (4.18) and (4.19) are the indefinite and Hilbert space inner products, respectively, of $H^2(\mathscr{D}_T)$, whereas the inner products on the right sides are the respective inner products of $\mathscr{D}_T$. We say that $k(\mu, \lambda)$ is a *reproducing kernel* for each of the inner products on $H^2(\mathscr{D}_T)$.

It is not obvious that the functions $k_\mu a$ $(a \in \mathscr{D}_{T^*}, |\mu|<1)$ are in $H(T)$; we show in Theorem 4.5 that this is in fact the case. Moreover, $k(\mu, \lambda)$ is a reproducing kernel for the indefinite inner product of $H(T)$:

$$[v, k_\mu a] = [v(\mu), a] \tag{4.20}$$

for every $v \in H(T)$, $a \in \mathscr{D}_{T^*}$, and for $|\mu|<1$. It is a consequence of (4.20) that the space of all finite linear combinations of functions of the form $k_\mu a$ $(a \in \mathscr{D}_{T^*}, |\mu|<1)$ is dense in $H(T)$. We also show in Theorem 4.5 that the inner product in $H(T)$ on the left side of (4.20) coincides with the inner product $[.,.]_\Sigma$.

In Section 5, we will find a reproducing kernel for the Hilbert space inner product of $H(T)$, i.e., a kernel k' such that

$$(v, k'_\mu a) = \big(v(\mu), a\big)$$

for every $v \in H(T)$, $a \in \mathscr{D}_{T^*}$; and for $|\mu|<1$.

Theorem 4.5. *The function* $k(\mu, \lambda)=(1-\lambda\bar{\mu})^{-1}$ *is a reproducing kernel for the indefinite inner product on* $H(T)$. *If* $a \in \mathscr{D}_{T^*}$ *and* $|\mu|<1$, *then we have*

$$k_\mu a = F(I-\bar{\mu}T)^{-1}Q_{T^*}a + \Theta_T k_\mu \Theta_T(\mu)^* a. \tag{4.21}$$

If u is a finite linear combination of functions of the form $k_\mu a$, where $a \in \mathscr{D}_{T^*}$ and $|\mu|<1$, then

$$[v, u] = [v, u]_\Sigma, \tag{4.22}$$

for all $v \in H(T)$.

Proof. From (3.5) we can derive the formula

$$I-\Theta_T(\lambda)\Theta_T(\mu)^* = (1-\lambda\bar{\mu})J_{T^*}Q_{T^*}(I-\lambda T^*)^{-1}(I-\bar{\mu}T)^{-1}Q_{T^*},$$

where λ and μ are in the open unit disc, and the adjoint $\Theta_T(\mu)^*$ is computed in the indefinite inner products of $\mathscr{D}_T$ and $\mathscr{D}_{T^*}$ (cf. [12], p. 238, and [8], Sec 4). Thus, for all $a \in \mathscr{D}_{T^*}$, we have

$$(1-\lambda\bar{\mu})^{-1}a = J_{T^*}Q_{T^*}(I-\lambda T^*)^{-1}(I-\bar{\mu}T)^{-1}Q_{T^*}a + \Theta_T(\lambda)(1-\lambda\bar{\mu})^{-1}\Theta_T(\mu)^* a,$$

and thus (4.21) is verified. Since $k_\mu \Theta_T(\mu)^* a \in H^2(\mathscr{D}_T)$, we have $k_\mu a \in H(T)$.

If we take $v=Fh+\Theta_T u \in H(T)$, then we obtain, using (4.21) and (4.2),

$$[v, k_\mu a] = \big(h, (I-\bar{\mu}T)^{-1}Q_{T^*}a\big)+[u, k_\mu \Theta_T(\mu)^* a] =$$
$$= \big(Q_{T^*}(I-\mu T^*)^{-1}h, a\big)+[u(\mu), \Theta_T(\mu)^* a],$$

using the reproducing property (4.18) in $H^2(\mathscr{D}_T)$. Rewriting the inner products in terms of the indefinite inner product of $\mathscr{D}_{T^*}$, we obtain

$$[v, k_\mu a] = [J_{T^*} Q_{T^*}(I-\mu T^*)^{-1} h + \Theta_T(\mu) u(\mu), a] = [v(\mu), a],$$

proving the reproducing property (4.20) for $H(T)$.

Finally, we note that, if

$$v(\lambda) = \sum_{n=0}^{\infty} \lambda^n v_n,$$

then

$$[v, k_\mu a]_\Sigma = \sum_{n=0}^{\infty} [v_n, \bar{\mu}^n a] = \sum_{n=0}^{\infty} [\mu^n v_n, a] = [v(\mu), a] = [v, k_\mu a].$$

Equation (4.22) then follows by linearity.

Although the indefinite inner product is given by $[.,.]_\Sigma$ on the dense subsets of $H(T)$ identified in Proposition 4.4 and Theorem 4.5, it does not necessarily apply on the whole space. As was noted above, it is possible to have $\mathscr{R}=\{0\}$ with $T \notin C_{.0}$; the following proposition shows that, in such a case, the inner product is not given by $[.,.]_\Sigma$ on the subspace $F\mathscr{H}$ of $H(T)$.

Proposition 4.6. *On the subspace $F\mathscr{H}$ of $H(T)$, the indefinite inner products $[.,.]$ and $[.,.]_\Sigma$ coincide if and only if $T \in C_{.0}$.*

Proof. Using (3.3) and (4.13), and the property $J_{T^*} Q_{T^*}^2 = I - TT^*$, we obtain

$$[Fh, Fk]_\Sigma = \sum_{n=0}^{\infty} [J_{T^*} Q_{T^*} T^{*n} h, J_{T^*} Q_{T^*} T^{*n} k] = \tag{4.23}$$

$$= \sum_{n=0}^{\infty} (T^n (I-TT^*) T^{*n} h, k) = (h, k) - \lim_{n\to\infty} (T^{*n} h, T^{*n} k)$$

whenever the limit exists. On the other hand, Proposition 4.1 gives $[Fh, Fk] = (h, k)$ for the inner product on $H(T)$. Thus, if $T \in C_{.0}$, the two inner products coincide. Conversely, if the inner products coincide, then, by putting $k=h$ in (4.23), we obtain $T^{*n} h \to 0$ for every $h \in \mathscr{H}$, i.e., $T \in C_{.0}$.

If we restrict ourselves to functions of the form $Fh + \Theta_T u$, where $h \in \mathscr{H}$ and u is a polynomial with values in $\mathscr{D}_T$, then we can show that, for $T \in C_{.0}$, the two indefinite inner products, given by (4.2) and (4.13), coincide. It follows immediately from the definition of $H(T)$ that the linear manifold of such functions is dense in $H(T)$.

Theorem 4.7. *Suppose $T \in C_{.0}$. Then the indefinite inner products $[.,.]$ and $[.,.]_\Sigma$ coincide on the dense linear manifold of $H(T)$ consisting of functions of the form $Fh + \Theta_T u$, with $h \in \mathscr{H}$ and u a polynomial with values in $\mathscr{D}_T$.*

Proof. Consider a function w of the form $w=Fh+\Theta_T u$, where $h\in\mathscr{H}$ and

$$u(\lambda) = \sum_{n=0}^{N} \lambda^n u_n,$$

for some $N\geqq 0$ and $u_n\in\mathscr{D}_T$ $(0\leqq n\leqq N)$. By the polarization identity, we can establish equality for the two inner products by showing that $[w, w]_\Sigma=[w, w]$, i.e., by showing that

$$[w, w]_\Sigma = \|h\|^2+[u, u] = \|h\|^2+\sum_{n=0}^{N}[u_n, u_n]. \tag{4.24}$$

We have already shown, in Proposition 4.6, that (4.24) is valid when $u=0$. Thus, to establish (4.24) it suffices to show that, if $h\in\mathscr{H}$, and if $u(\lambda)=\lambda^n a$ and $v(\lambda)=\lambda^m b$, for $a, b\in\mathscr{D}_T$ and $0\leqq m<n$, then

$$[Fh, \Theta_T u]_\Sigma = 0, \tag{4.25}$$

$$[\Theta_T u, \Theta_T v]_\Sigma = 0, \tag{4.26}$$

and

$$[\Theta_T u, \Theta_T u]_\Sigma = [a, a]. \tag{4.27}$$

The definitions (3.3) and (3.5) of Fh and $\Theta_T(\lambda)$, together with the definition (4.13) of the inner product, give us

$$\begin{aligned}
[Fh, \Theta_T u]_\Sigma &= \sum_{k=0}^{\infty}[J_{T^*}Q_{T^*}T^{*n+k}h, \Theta_k a] = \\
&= -[J_{T^*}\Theta_{T^*}T^{*n}h, TJ_T a]+\sum_{k=1}^{\infty}[J_{T^*}Q_{T^*}T^{*n+k}h, J_{T^*}Q_{T^*}T^{*k-1}J_T Q_T a] = \\
&= -(J_T Q_T T^{*n+1}h, a)+\sum_{k=1}^{\infty}(J_T Q_T T^{k-1}(I-TT^*)T^{*n+k}h, a) = \\
&= -\lim_{k\to\infty}(T^{*n+k+1}h, T^{*k}J_T Q_T a) = 0,
\end{aligned}$$

since $T\in C_{\cdot 0}$. This proves (4.25); to prove (4.26), note that we have, for $m<n$,

$$\begin{aligned}
[\Theta_T u, \Theta_T v]_\Sigma &= \sum_{k=0}^{\infty}[\Theta_k a, \Theta_{k+n-m}b] = \\
&= -[TJ_T a, J_{T^*}Q_{T^*}T^{*n-m-1}J_T Q_T b]+ \\
&+\sum_{k=1}^{\infty}[J_{T^*}Q_{T^*}T^{*k-1}J_T Q_T a, J_{T^*}Q_{T^*}T^{*k+n-m-1}J_T Q_T b] = \\
&= -(T^{n-m}J_T Q_T a, J_T Q_T b)+\sum_{k=1}^{\infty}(T^{k+n-m-1}(I-TT^*)T^{*k-1}J_T Q_T a, J_T Q_T b) = \\
&= -\lim_{k\to\infty}(T^{*k}J_T Q_T a, T^{*k+n-m}J_T Q_T b) = 0.
\end{aligned}$$

The remaining identity (4.27) follows immediately from (3.9).

Corollary 4.8. *Suppose* $T \in C_{\cdot 0}$. *Then the indefinite inner products* $[.,.]$ *and* $[.,.]_\Sigma$ *coincide on the dense linear manifold of* $H(T)$ *consisting of finite linear combinations of functions of the form* $\mathbf{U}^n Fh$, *where* $h \in \mathscr{H}$ *and* $n \geqq 0$.

Proof. We can obtain from (4.10) the formula

$$\mathbf{U}^n Fh = FT^n h + \Theta_T \sum_{k=0}^{n-1} \mathbf{U}^k Q_T T^{n-k-1} h, \tag{4.28}$$

showing that Theorem 4.7 applies to functions in the manifold consisting of all linear combinations of functions of the form $\mathbf{U}^n Fh$. The fact that this manifold is dense in $H(T)$ can easily be proved by noting that only the zero function in $H(T)$ can be orthogonal to all functions of the form (4.28).

5. Reproducing kernels

We assume, as in previous sections, that T is an operator with spectrum in the closed unit disc and with trivial residual space. In Section 3, we represented the range of the Fourier representation Φ_* in the form

$$H(T) = F\mathscr{H} + \Theta_T H^2(\mathscr{D}_T),$$

and the operator T was used explicitly in the construction of this space. By contrast, when the characteristic function is bounded, we have $H(T) = H^2(\mathscr{D}_{T^*})$ ([12], [5]), and thus a knowledge of only the space $\mathscr{D}_{T^*}$ suffices to construct $H(T)$. In this section, we show how $H(T)$ can be described in terms of the characteristic function Θ_T, without explicit reference to T, and use this in the following sections to obtain a functional model in terms of Θ_T.

The space $\Theta_T H^2(\mathscr{D}_T)$ already has a description in terms of Θ_T alone, since a knowledge of $\mathscr{D}_T$, the domain space of Θ_T, is all that is required to describe the space $H^2(\mathscr{D}_T)$. However, the description of $F\mathscr{H}$ in terms of Θ_T is not so immediate.

By Theorem 4.5, the space $H(T)$ contains all functions of the form $k_\mu a$, where

$$[k_\mu a](\lambda) = k(\mu, \lambda) a = (1 - \lambda \bar{\mu})^{-1} a, \quad a \in \mathscr{D}_{T^*}, \ |\mu| < 1,$$

and $k(\mu, \lambda)$ is a reproducing kernel for the indefinite inner product of $H(T)$. If we consider the orthogonal projection of $k_\mu a$ onto $F\mathscr{H}$, we should obtain a reproducing kernel for the indefinite inner product of $F\mathscr{H}$. In Theorem 5.1 below, we show that the kernel so obtained is

$$K(\mu, \lambda) = (1-\lambda\bar{\mu})^{-1}(I-\Theta_T(\lambda)\Theta_T(\mu)^*). \tag{5.1}$$

We prove the following reproducing property: if

$$[K_\mu a](\lambda) = K(\mu, \lambda)a \tag{5.2}$$

for $a \in \mathscr{D}_{T^*}$ and $|\mu|<1$, then $K_\mu a \in F\mathscr{H}$ and

$$[Fh, K_\mu a] = [(Fh)(\mu), a] \tag{5.3}$$

for all $h \in \mathscr{H}$.

We also show in Theorem 5.1 that the function

$$K(\mu, \lambda) = K(\mu, \lambda) J_{T^*} \tag{5.4}$$

is a reproducing kernel for the Hilbert space inner product on $F\mathscr{H}$: if

$$[K'_\mu a](\lambda) = K'(\mu, \lambda)a \tag{5.5}$$

for $a \in \mathscr{D}_{T^*}$ and $|\mu|<1$, then $K'_\mu a \in F\mathscr{H}$ and

$$(Fh, K'_\mu a) = ([Fh](\mu), a)$$

for all $h \in \mathscr{H}$. Note that the indefinite and Hilbert space inner products coincide on $F\mathscr{H}$; the only reason that separate kernels are needed for the two inner products is that they don't coincide on $\mathscr{D}_{T^*}$.

Theorem 5.1. *The subspace $F\mathscr{H}$ of $H(T)$ is the closed linear span of functions of the form $K_\mu a$ (defined by (5.2)), where $a \in \mathscr{D}_{T^*}$ and $|\mu|<1$. The functions $K(\mu, \lambda)$ and $K'(\mu, \lambda)$, defined by* (5.1) *and* (5.4), *are reproducing kernels for the indefinite and the Hilbert space inner products, respectively, on $F\mathscr{H}$.*

Proof. From the representation of the function $k_\mu a$ given by (4.21), we obtain

$$[F(I-\bar{\mu}T)^{-1}Q_{T^*}a](\lambda) = (1-\lambda\bar{\mu})^{-1}(I-\Theta_T(\lambda)\Theta_T(\mu)^*)a = [K_\mu a](\lambda),$$

showing that the functions $K_\mu a$ are in $F\mathscr{H}$. By Proposition 4.1, the inner products on $F\mathscr{H}$ are given by

$$[Fh, Fh'] = (Fh, Fh') = (h, h'),$$

for all $h, h' \in \mathscr{H}$. For the functions given by (5.2) and (5.5), we therefore have, for all $h \in \mathscr{H}$,

$$[Fh, K_\mu a] = (h, (I-\bar{\mu}T)^{-1}Q_{T^*}a) = [J_{T^*}Q_{T^*}(I-\mu T^*)^{-1}h, a] = [(Fh)(\mu), a]$$

and

$$(Fh, K'_\mu a) = (h, (I-\bar{\mu}T)^{-1}Q_{T^*}J_{T^*}a) = (J_{T^*}Q_{T^*}(I-\mu T^*)^{-1}h, a) = ([Fh](\mu), a).$$

Therefore, $K(\mu, \lambda)$ and $K'(\mu, \lambda)$ are reproducing kernels for the two inner products. Since the indefinite inner product of $\mathscr{D}_{T^*}$ is nondegenerate, (5.3) also shows that only the zero function Fh is orthogonal to every function $K_\mu a$, with $a \in \mathscr{D}_{T^*}$ and $|\mu| < 1$, and thus the space of linear combinations of such functions is dense in $F\mathscr{H}$.

The inner products on $\Theta_T H^2(\mathscr{D}_T)$ can also be given in terms of reproducing kernels. Recall that $\Theta_T(\mu)^*$ denotes the adjoint of $\Theta_T(\mu)$ with respect to the indefinite inner products on $\mathscr{D}_T$ and $\mathscr{D}_{T^*}$. We will denote by $\Theta_T(\mu)^{(*)}$ the adjoint of $\Theta_T(\mu)$ with respect to the Hilbert space inner products on $\mathscr{D}_T$ and $\mathscr{D}_{T^*}$.

Theorem 5.2. *The function*

$$L(\mu, \lambda) = (1 - \lambda\bar{\mu})^{-1} \Theta_T(\lambda) \Theta_T(\mu)^*$$

is reproducing for the indefinite inner product, and the function

$$L'(\mu, \lambda) = (1 - \lambda\bar{\mu})^{-1} \Theta_T(\lambda) \Theta_T(\mu)^{(*)}$$

is reproducing for the Hilbert space inner product on $\Theta_T H^2(\mathscr{D}_T)$.

Proof. If $L_\mu a$ and $L'_\mu a$ are defined for $a \in \mathscr{D}_{T^*}$ and $|\mu| < 1$ by $[L_\mu a](\lambda) = L(\mu, \lambda) a$ and $[L'_\mu a](\lambda) = L'(\mu, \lambda) a$, then, clearly, $L_\mu a \in \Theta_T H^2(\mathscr{D}_T)$ and $L'_\mu a \in \Theta_T H^2(\mathscr{D}_T)$.

For every $u \in H^2(\mathscr{D}_T)$ we have, using (4.2) for the indefinite inner product on $\Theta_T H^2(\mathscr{D}_T)$, and the reproducing property (4.18) of $k(\mu, \lambda)$ on $H^2(\mathscr{D}_T)$,

$$[\Theta_T u, L_\mu a] = [u, k_\mu \Theta_T(\mu)^* a] = [u(\mu), \Theta_T(\mu)^* a] = [\Theta_T(\mu) u(\mu), a],$$

proving the reproducing property for the indefinite inner product. Similarly,

$$(\Theta_T u, L'_\mu a) = (\Theta_T(\mu) u(\mu), a),$$

proving the reproducing property for the Hilbert space inner product.

Note that the reproducing kernel $k(\mu, \lambda)$ for the indefinite inner product of $H(T)$ can be obtained as the sum of $K(\mu, \lambda)$ and $L(\mu, \lambda)$. We can obtain a reproducing kernel for the Hilbert space inner product of $H(T)$ by considering

$$k'(\mu, \lambda) = K'(\mu, \lambda) + L'(\mu, \lambda). \tag{5.6}$$

Theorem 5.3. *The function* $k'(\mu, \lambda)$, *defined by* (5.6), *is a reproducing kernel for the Hilbert space inner product of* $H(T)$.

Proof. This follows immediately from Theorems 5.1 and 5.2, and the fact that the spaces $F\mathscr{H}$ and $\Theta_T H^2(\mathscr{D}_T)$ are orthogonal complements in the Hilbert space inner product of $H(T)$.

6. The space $H(T)$

In the preceding sections, the space $H(T)$ was described for an arbitrary operator with trivial residual space. In this section, we obtain a description of a space $H(\Theta)$, for an operator valued analytic function Θ. The function Θ will be assumed to satisfy conditions that are known to be valid for the characteristic function of a $C_{\cdot 0}$ operator. These assumptions will be sufficient to guarantee that Θ is the characteristic function of a completely non-unitary operator T with trivial residual space, and we will then have $H(\Theta)=H(T)$.

Throughout this section, we suppose that Θ is an operator valued analytic function, defined on the open unit disc, and taking values that are operators from a Krein space $\mathscr{D}$ to a Krein space $\mathscr{D}_*$. For $|\lambda|<1$ we can write

$$(6.1)\qquad \Theta(\lambda) = \sum_{n=0}^{\infty} \lambda^n \Theta_n,$$

where, for each $n \geqq 0$, Θ_n is a bounded operator from $\mathscr{D}$ to $\mathscr{D}_*$.

We assume that Θ is fundamentally reducible, i.e., that there are fundamental symmetries on $\mathscr{D}$ and $\mathscr{D}_*$ commuting with $\Theta(0)^*\Theta(0)$ and $\Theta(0)\Theta(0)^*$, respectively (see [8]). We also assume that Θ is purely contractive, i.e. if $|\lambda|<1$, then

$$[\Theta(\lambda)a, \Theta(\lambda)a] < [a, a] \quad \text{for} \quad a \in \mathscr{D},\ a \neq 0,$$

and

$$[\Theta(\lambda)^* b, \Theta(\lambda)^* b] < [b, b] \quad \text{for} \quad b \in \mathscr{D}_*,\ b \neq 0.$$

As usual, we are using $[.,.]$ to denote the indefinite inner products on $\mathscr{D}$ and $\mathscr{D}_*$, and $\Theta(\lambda)^*$ denotes the adjoint of $\Theta(\lambda)$ with respect to these inner products.

It follows from the above hypotheses that Θ is the characteristic function of a uniquely determined completely non-unitary operator T and, conversely, the characteristic function of any completely non-unitary operator satisfies these hypotheses (see [8], [1]). Since Θ is analytic in the open unit disc, it also follows from [1] that T has spectrum in the closed unit disc. We will also be assuming that Θ satisfies the additional condition

$$(6.2)\qquad \sum_{n=0}^{\infty} \Theta_n^* \Theta_n = I,$$

in the strong operator topology, where the operators Θ_n are given by (6.1). It was shown previously, in (3.9), that Θ satisfies (6.2) if it is the characteristic function of a $C_{\cdot 0}$ operator. It is not known if T is necessarily in $C_{\cdot 0}$ when Θ satisfies (6.2).

In this paper, we will be constructing a different functional model for T than that given by Ball in [1], but we will be appealing to Ball's model to be able to assert that $\Theta = \Theta_T$ for some completely non-unitary operator T acting on a Hilbert space

$\mathscr{H}$. The assumption (6.2) on Θ implies the condition of trivial residual space for T, which we considered earlier. The proof of this, in Theorem 6.1 below, closely resembles the proof, in [9], Theorems 4.2 and 5.5, of the fact that $\mathscr{R}=\{0\}$ for a $C_{\cdot 0}$ operator.

We use the same notation below as we have used previously. In particular, F is the function given by (3.3).

Theorem 6.1. *Suppose Θ is a fundamentally reducible, purely contractive analytic function, satisfying the condition* (6.2), *and let T be the completely non-unitary operator such that $\Theta=\Theta_T$. If $Fh+\Theta_T u=0$ for some $h\in\mathscr{H}$ and $u\in H^2(\mathscr{D})$, then $h=0$ and $u=0$. T has trivial residual space: $\mathscr{R}=\{0\}$.*

Proof. By Theorem 4.2, it suffices to prove the first part: if $Fh+\Theta_T u=0$ for $h\in\mathscr{H}$ and $u\in H^2(\mathscr{D}_T)$, then $h=0$ and $u=0$.

If we assume that $Fh+\Theta_T u=0$, then we obtain, from the nth coefficient in the Taylor series expansion of $Fh+\Theta_T u$,

$$J_{T^*}Q_{T^*}T^{*n}h+\sum_{k=0}^{n}\Theta_k u_{n-k}=0, \tag{6.3}$$

where

$$u(\lambda)=\sum_{k=0}^{\infty}\lambda^k u_k.$$

Define a sequence $\{h_n\}_{n\geqq 0}$ in $\mathscr{H}$ by

$$h_n=T^{*n}h+\sum_{k=1}^{n}T^{*k-1}J_TQ_Tu_{n-k},$$

for $n\geqq 0$. Then we have

$$h_0=h, \tag{6.4}$$

and, for each $n\geqq 0$,

$$\begin{aligned}h_n-Th_{n+1}&=(I-TT^*)T^{*n}h-TJ_TQ_Tu_n+\sum_{k=1}^{n}(I-TT^*)T^{*k-1}J_TQ_Tu_{n-k}=\\&=Q_{T^*}\big[J_{T^*}Q_{T^*}T^{*n}h-TJ_Tu_n+\sum_{k=1}^{n}J_{T^*}Q_{T^*}T^{*k-1}J_TQ_Tu_{n-k}\big]=\\&=Q_{T^*}\big[J_{T^*}Q_{T^*}T^{*n}h+\sum_{k=0}^{n}\Theta_ku_{n-k}\big]=0\end{aligned}$$

by (6.3). Thus, for all $n\geqq 0$, we have

$$Th_{n+1}=h_n, \tag{6.5}$$

and by induction we obtain, for $0\leqq n\leqq N$,

$$T^{N-n}h_n=h_n. \tag{6.6}$$

We also have, for all $n \geqq 0$,

$$h_{n+1} = T^{*n+1}h + \sum_{k=1}^{n+1} T^{*k-1} J_T Q_T u_{n+1-k} = \tag{6.7}$$

$$= T^{*n+1}h + \sum_{k=0}^{n} T^{*k} J_T Q_T u_{n-k} = T^* h_n + J_T Q_T u_n.$$

Thus, using (6.7) and (6.5), we get

$$J_T Q_T u_n = h_{n+1} - T^* h_n = (I - T^*T) h_{n+1} = J_T Q_T (Q_T h_{n+1}).$$

Since $J_T Q_T$ is injective on $\mathscr{D}_T$, it follows that

$$u_n = Q_T h_{n+1} \tag{6.8}$$

for all $n \geqq 0$.

Since $u \in H^2(\mathscr{D}_T)$, we can write for the indefinite inner product (using (6.8))

$$[u, u] = \sum_{n=0}^{\infty} [u_n, u_n] = \lim_{N\to\infty} \sum_{n=0}^{N-1} [Q_T h_{n+1}, Q_T h_{n+1}] =$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} [Q_T h_n, Q_T h_n] = \lim_{N\to\infty} \sum_{n=1}^{N} [Q_T T^{N-n} h_N, Q_T T^{N-n} h_N],$$

by (6.6). Thus we obtain the telescoping series

$$[u, u] = \lim_{N\to\infty} \sum_{n=1}^{N} \left(T^{*N-n}(I - T^*T) T^{N-n} h_N, h_N\right) = \tag{6.9}$$

$$= \lim_{N\to\infty} (\|h_N\|^2 - \|T^N h_N\|^2) = \lim_{N\to\infty} \|h_N\|^2 - \|h_0\|^2,$$

by (6.6) again. It follows, from the existence of the limit in (6.9), that the sequence $\{h_n\}_{n\geqq 0}$ must be bounded.

The condition (6.2) on Θ is equivalent to

$$\lim_{n\to\infty} Q_T T^n T^{*n} J_T Q_T = 0,$$

in the strong operator topology (cf. (3.9)). Therefore, for every $k \in \mathscr{H}$, we have

$$\|T^{*n} Q_T k\|^2 = \left(Q_T T^n T^{*n} J_T Q_T (J_T k), k\right) \to 0 \quad \text{as} \quad n \to \infty.$$

Using the boundedness of the sequence $\{h_n\}_{n\geqq 0}$ and property (6.6), we can conclude that

$$(k, Q_T h_n) = (Q_T k, T^{N-n} h_N) = (T^{*N-n} Q_T k, h_N) \to 0 \quad \text{as} \quad N \to \infty.$$

Consequently, $Q_T h_n = 0$ for each $n \geqq 0$; by (6.8) and (6.4), this implies that $u = 0$ and that $Q_T h = 0$.

From (6.4) and (6.6) we can also conclude that, for each $k\in\mathscr{H}$ and $n\geqq 0$,

$$(k, Q_T T^n h) = (k, Q_T T^{n+N} h_N) = (T^{*n+N} Q_T k, h_N) \to 0 \quad \text{as} \quad N\to\infty,$$

and thus

$$Q_T T^n h = 0 \quad \text{for all} \quad n\geqq 0. \tag{6.10}$$

Since $u=0$, we have $Fh=0$, and this implies that

$$Q_{T^*} T^{*n} h = 0 \quad \text{for all} \quad n\geqq 0. \tag{6.11}$$

We complete the proof by showing that the two conditions (6.10) and (6.11) together imply that $h=0$. The subspace $\mathscr{H}_0$ of all vectors $h\in\mathscr{H}$ satisfying (6.10) and (6.11) is invariant for T; this follows from the relations $Q_{T^*}Th=TQ_T h$ and

$$Q_{T^*}T^{*n}Th = Q_{T^*}T^{*n-1}(T^*Th) = Q_{T^*}T^{*n-1}h \quad (n\geqq 1)$$

when $Q_T h=0$. By symmetry, $\mathscr{H}_0$ is invariant for T^* as well. The relations $Q_T h=0$ and $Q_{T^*}h=0$ imply that $\mathscr{H}_0$ reduces T to a unitary operator; since T is assumed to be completely non-unitary, we have $\mathscr{H}_0=\{0\}$.

The space $F\mathscr{H}$ will be modelled by following the representation given in Theorem 5.1. We define, as before,

$$K(\mu,\lambda) = (1-\lambda\bar{\mu})^{-1}(I-\Theta(\lambda)\Theta(\mu)^*)$$

and

$$K_\mu(\lambda) = K(\mu,\lambda).$$

Consider the space $\mathbf{H}_0$ of all finite linear combinations of functions of the form $K_\mu a$, where $a\in\mathscr{D}_*$ and $|\mu|<1$. We impose on $\mathbf{H}_0$ an inner product $[.,.]$ by means of the formula (5.3):

$$[u, K_\mu a] = [u(\mu), a],$$

for all $u\in\mathbf{H}_0$ and for all $a\in\mathscr{D}_*$ and $|\mu|<1$. Part of the proof of Theorem 1 of [8] shows that this inner product is positive definite. If $\mathbf{H}$ denotes the completion of the space $\mathbf{H}_0$ to a Hilbert space, then standard reproducing kernel arguments can be used to show that $\mathbf{H}$ can be identified with a space of functions analytic in the open unit disc. Since $\mathscr{R}=\{0\}$, Theorem 5.1 shows that $\mathbf{H}=F\mathscr{H}$.

We define the space $H(\Theta)$ as

$$H(\Theta) = \mathbf{H}+\Theta H^2(\mathscr{D}).$$

Since $\mathbf{H}$ can be identified with $F\mathscr{H}$, Theorem 6.1 implies that every function in $H(\Theta)$ has a unique representation in the form $h+\Theta u$, with $h\in\mathbf{H}$ and $u\in H^2(\mathscr{D})$. Suppose $v=h+\Theta u$ and $v'=h'+\Theta u'$ are two functions in $H(\Theta)$; we can define indefinite and Hilbert space inner products on $H(\Theta)$ by

$$[v, v'] = (h, h')+[u, u'] \tag{6.12}$$

and

(6.13) $$(v, v') = (h, h') + (u, u').$$

The first of the inner products on the right sides of (6.12) and (6.13) is the inner product on the Hilbert space $\mathbf{H}$; the second of the inner products is the indefinite inner product (in (6.12)) and the Hilbert space inner product (in (6.13)) on $H^2(\mathscr{D})$. With these inner products, $H(\Theta)$ is a Krein space, with fundamental symmetry J given by

$$J(h+\Theta u) = h+\Theta(Ju) \quad (h \in \mathbf{H}, u \in H^2(\mathscr{D})).$$

A comparison of the constructions of the spaces $H(\Theta)$ and $H(T)$ shows that $H(\Theta) = H(T)$.

Note that we have constructed $H(\Theta)$ in terms of Θ alone; we needed to use the fact that $\Theta = \Theta_T$ for some operator T only to prove some properties of $H(\Theta)$ from the assumptions on Θ. It would be more desirable to be able to construct the space $H(\Theta)$ without any reference to the underlying operator; the stumbling block is finding a direct product of the uniqueness of the representation $h+\Theta u$ for $h \in \mathbf{H}$ and $u \in H^2(\mathscr{D})$.

7. Functional model for an operator

In the first part of this section, we assume that T is an operator with spectrum in the closed unit disc and with trivial residual space. Such an operator is automatically completely non-unitary, since a subspace of $\mathscr{H}$ which reduces T to a unitary operator is in the residual space (see [9], Theorem 3.1). We present here a model for this operator, based on the function space $H(T)$ constructed earlier. We will finish the section by presenting a model in terms of an operator valued analytic function Θ.

Let $\mathbf{K}_+ = H(T)$, and let $\mathbf{U}$ denote multiplication by the independent variable, as in Section 4. Then the Fourier representation Φ_* is a unitary map from $\mathscr{K}_+$ onto $\mathbf{K}_+$, preserving both the indefinite and the Hilbert space inner products. The subspace $\mathscr{H}$ of $\mathscr{K}_+$ is identified with the subspace $\mathbf{H}$ of $\mathbf{K}_+$, defined as the orthogonal complement of $\Theta_T H^2(\mathscr{D}_T)$ in $\mathbf{K}_+$:

(7.1) $$\mathbf{H} = \mathbf{K}_+ \cap [\Theta_T H^2(\mathscr{D}_T)]^\perp.$$

The representation of $H(T)$ in the form $F\mathscr{H} + \Theta_T H^2(\mathscr{D}_T)$, and the forms (4.2) and (4.3) of the inner products on $H(T)$, show that either of the two inner products could be used for the orthogonal complement in (7.1), and that Φ_* maps $\mathscr{H}$ onto $\mathbf{H}$.

If we define $U_+ = U|\mathscr{K}_+$, then we have

$$T^* = U_+^*|\mathscr{H}$$

(cf. [12]). It follows immediately that, if we define

$$\mathbf{T}^* = \mathbf{U}^*|\mathbf{H},$$

then we have

$$\Phi_* Th = \mathbf{T}\Phi_* h$$

for all $h \in \mathscr{H}$.

The following theorem summarizes the main properties of the model, which is based on the Sz.-Nagy and Foiaş model (see [12] and [10]) of an operator. We present a model only for the part of the dilation on $\mathscr{K}_+$; the remainder of the space could be modelled very simply by including functions of the form

$$v(\lambda) = \sum_{n=-\infty}^{-1} \lambda^n v_n,$$

with square summable coefficients $v_n \in \mathscr{D}_{T^*}$, but notational convenience would be sacrificed.

We use the function space $H(T)$ in place of the space $H^2(\mathscr{D}_{T^*})$ of the Sz.-Nagy and Foiaş model. The model is simplified by the fact that we are working with operators for which $\mathscr{R} = \{0\}$.

Theorem 7.1. *The Fourier representation Φ_* of $M_+(\mathscr{L}_*)$ is a unitary operator from $\mathscr{K}_+$ onto $\mathbf{K}_+$, preserving both the indefinite and the Hilbert space inner products. If U is the Davis dilation of T, restricted to the subspace $\mathscr{K}_+$, then $\Phi_* U = \mathbf{U}\Phi_*$.*

The subspace $\mathscr{H}$ of $\mathscr{K}_+$ is mapped by Φ onto the subspace $\mathbf{H}$ of $\mathbf{K}_+$, defined by (7.1), *and the indefinite and Hilbert space inner products coincide on* $\mathbf{H}$. *If* $\mathbf{T}$ *is the operator on* $\mathbf{H}$ *whose adjoint is defined by*

$$\mathbf{T}^* u = \mathbf{U}^* u,$$

for $u \in \mathbf{H}$, then we have, for all $h \in \mathscr{H}$, $\Phi_* Th = \mathbf{T}\Phi_* h$.

When the characteristic function Θ_T is uniformly bounded on the open unit disc, the space $H^2(\mathscr{D}_{T^*})$ can be used as the range of Φ_* [5]. In that case Φ_* is bounded, with bounded inverse, but does not preserve the Hilbert space inner products. Since the shift on $H^2(\mathscr{D}_{T^*})$ is an isometry, in the Hilbert space inner product, the analogue of Theorem 7.1 gives the result that, when Θ_T is bounded, U is similar to a unitary operator on a Hilbert space ([11], and [9], Theorem 7.2).

When $H(T)$ is used as the range of Φ_*, Theorem 7.1 above shows that the Hilbert space inner product is preserved by Φ_*. We lose, however, the property that the shift is an isometry in the Hilbert space inner product: the operator $\mathbf{U}$ on $H(T)$ is a shift in the Kreins space sense, preserving the indefinite inner product, but not necessarily the Hilbert space inner product. Indeed, $\mathbf{U}$ need not be power bounded when $T \in C_{\cdot 0}$, and hence need not be similar to a unitary operator on a Hilbert space.

Example 7.2. Let T be the adjoint of the operator defined in Example 3.1, so that $T \in C_{\cdot 0}$. Suppose $u = Fx \in H(T)$, for the vector $x \in \mathscr{H}$ used in Example 3.1. Then, by (4.28), we have

$$[\mathbf{U}^n u](\lambda) = \lambda^n u(\lambda) = [FT^n x](\lambda) + \Theta_T(\lambda) \sum_{k=0}^{n-1} \lambda^{n-k-1} Q_T T^k x$$

and thus

$$\|\mathbf{U}^n u\|^2 = \|T^n x\|^2 + \sum_{k=0}^{n-1} \|Q_T T^k x\|^2. \tag{7.2}$$

Since we are working with the adjoint of the operator in Example 3.1, (3.4) shows that the sequence $\{J_T Q_T T^k x\}_{k \geq 0}$ is not square summable. Since J_T is unitary, the sequence $\{Q_T T^k x\}_{k \geq 0}$ is not square summable, and so, by (7.2), $\mathbf{U}$ is not power bounded.

We finish the section by assuming that $\Theta(\lambda)$: $\mathscr{D} \to \mathscr{D}_*$ is a fundamentally reducible, purely contractive analytic function, which satisfies condition (6.2), We present here a model for an operator having Θ as its characteristic function, based on the function space $H(\Theta)$ constructed in the preceding section. We know, from the previous section, that Θ is the characteristic function of a unique completely non-unitary operator T, acting on a Hilbert space $\mathscr{H}$, with spectrum in the closed unit disc, and with trivial residual space. Thus, we can describe the model directly, using the above results and the fact that $H(T) = H(\Theta)$. Note that the space $\mathbf{H}$, defined by (7.1), is the same as the space $\mathbf{H}$ defined in Section 6.

Theorem 7.3. *Suppose* $\Theta(\lambda)$: $\mathscr{D} \to \mathscr{D}_*$ *is a fundamentally reducible, purely contractive analytic function, which satisfies condition* (6.2), *Define* $\mathbf{K}_+ = H(\Theta)$, *and*

$$\mathbf{H} = \mathbf{K}_+ \cap [\Theta H^2(\mathscr{D})]^\perp,$$

where the orthogonal complement is taken in either of the two inner products of $H(\Theta)$. *Then* $\mathbf{H}$ *is a Hilbert space, and the operator* $\mathbf{T}$ *on* $\mathbf{H}$, *defined by*

$$\mathbf{T}^* u = \mathbf{U}^* u,$$

for $u \in \mathbf{H}$, *has characteristic function which coincides with* Θ.

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